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ON DOMAIN DECOMPOSITION METHODS FOR
SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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ON DOMAIN DECOMPOSITION METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

A domain decomposition method for solving partial differential equations is described. The conditions on interfaces will all be of Dirichlet type and obtained by the boundary element method using very few (less than 10) unknowns.

1. INTRODUCTION

Domain decomposition methods are based, as the name suggests, on subdivision of the domain into several subdomains and solving the problem on several subdomains in parallel. These methods are becoming the focus of research on numerical methods for the solution of partial differential equations (Widlund [34]). The methods can be regarded as divide-and-conquer algorithms since the problems on the subregions can be solved by well known techniques. The interactions between the solutions on the subdomains lead to an iterative procedure. When the number of subdomains is large one can improve the convergence of this iterative procedure by using a coarse grid to obtain starting values for the solution on interfaces. In this respect, the methods are similar to multigrid methods. The crucial point for domain decomposition schemes is how to pass information from one domain to other processors. Two different approaches were followed in the literature (see [2], [8], [10], [11], [16], [30], and references there). The first approach is based on decomposition of the domain into contiguous regions (see [13], [20]–[23], [25], [26], [33], and others). The second is based on having overlapping regions (Schwarz alternating method [9], [12], [18], [19], [28], [29], [32], [35], and others).

The main difficulty of such parallel techniques is in the initial assignment of values to the interfaces between domains. The more accurate such values are, the faster the convergence. As we mentioned earlier, one can use the solution on a coarse mesh (as in multigrid). Here we suggest the use of boundary element methods to approximate the solution at interfaces.

Boundary element methods were developed by Brebbia [5] and others. These methods are now widely used in various linear and nonlinear problems. Several papers ([1], [4], [6]–[8], [14], [17], [24], [25], [31]) are listed here. The list is by no means exhaustive.

In the next section, we introduce the problem and describe the domain decomposition method to be used. Section 3 will describe the boundary element method and its use for the approximation of interface values. In section 4, we give the details of the algorithm on Intel IPSC/2 Hypercube.

2. DOMAIN DECOMPOSITION

Consider the following elliptic problem

$$-\Delta u = f \qquad \text{in} \qquad \Omega \tag{1}$$

$$u = g \qquad \text{on} \qquad \partial\Omega \tag{2}$$

where Ω is the L shaped domain (Figure 1). The domain is divided into M subdomains Ω_i. The size of each subdomain will be such that the work is balanced among the processors. The subdomains are "colored" or numbered. Each subdomain borders subdomains of different colors (or numbers).

If we have boundary conditions for all M_1 subdomains numbered 1, one can assign these domains to the p processors for independent solution. Once the solution is obtained, the next set can be taken. If the domains overlap, there will be no need to transfer data; otherwise data is transferred to the neighboring subregions.

Remarks:

1. Boundary conditions on interfaces will be obtained following the recipe in the next section.

2. If $p < M_1$, then we solve for the first p subdomains numbered 1, transfer information if necessary and take the next p subdomains. If at some step one is left with less than

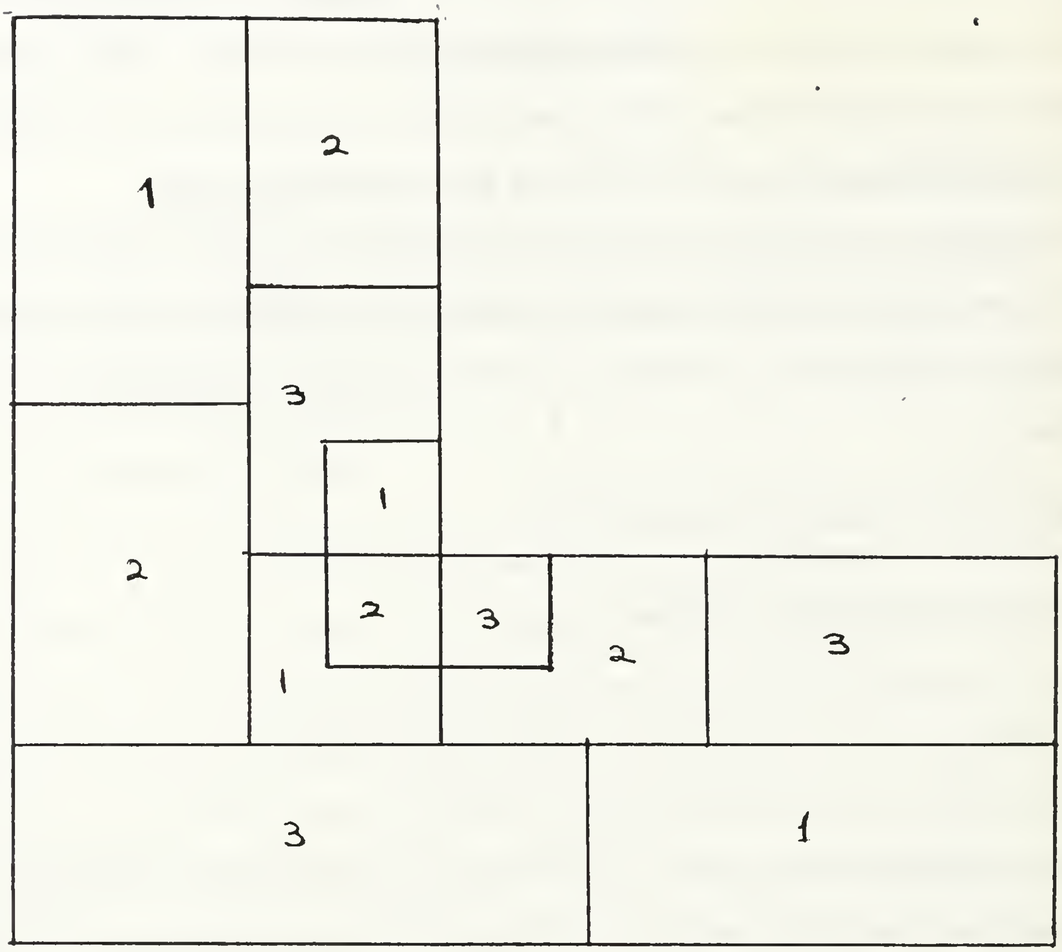

Figure 1:

p subregions of the same "color", the other processors can start the next "color".

3. BOUNDARY ELEMENT METHODS

"Boundary elements (Brebbia et al. [6]) have emerged as a powerful alternative to finite elements particularly in cases where better accuracy is required due to problems such as stress concentration or where the domain extends to infinity. The most important feature of boundary elements, however, is that it only requires discretization of the surface rather than the volume" (Brebbia and Dominguez [7]). Several examples were given there to the advantage of boundary elements.

The starting boundary integral equation required by the method can be deduced in a simple way based, for example, on weighted residuals (see e.g., Brebbia and Dominguez [7]).

It was shown [7] that the problem

$$\nabla^2 u = 0 \qquad \text{in} \qquad \Omega \tag{3}$$

$$u = \overline{u} \qquad \text{on} \qquad \Gamma_D \tag{4}$$

$$q = \frac{\partial u}{\partial n} = \overline{q} \qquad \text{on} \qquad \Gamma_N \tag{5}$$

where n is the outward normal to the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ is equivalent to

$$\int_\Omega (\nabla^2 u^*) u d\Omega = -\int_{\Gamma_N} \overline{q} u^* d\Gamma - \int_{\Gamma_D} q u^* d\Gamma + \int_{\Gamma_N} u q^* d\Gamma + \int_{\Gamma_D} \overline{u} q^* d\Gamma, \tag{6}$$

where u^* is a weight function with normal derivative q^* on the boundary. The boundary integral equation is

$$\frac{1}{2} u^i + \int_\Gamma u q^* d\Gamma = \int_\Gamma q u^* d\Gamma \tag{7}$$

where u^i is the value $u(x^i)$ and x^i is a boundary point. The numerical solution of the integral equation is accomplished by dividing Γ into N pieces Γ_i (Γ_D into N_1 pieces and Γ_N into N_2 pieces) and solving the system

$$\frac{1}{2} u^i + \sum_{j=1}^{N} \hat{H}^{ij} u^j = \sum_{j=1}^{N} G^{ij} q^j \qquad i = 1, 2, \ldots, N \tag{8}$$

where

$$\hat{H}^{ij} = \int_{\Gamma_j} q^* d\Gamma \tag{9}$$

$$G^{ij} = \int_{\Gamma_j} u^* d\Gamma. \tag{10}$$

In matrix form

$$HU = GQ \tag{11}$$

where

$$H^{ij} = \begin{cases} \hat{H}^{ij} & i \neq j \\ \hat{H}^{ij} + \frac{1}{2} & i = j \end{cases} \tag{12}$$

Note that N_1 values of u and N_2 values of q are known hence there are only N unknowns. One can rearrange the system so that X will contain the unknowns and solve

$$AX = F. \tag{13}$$

F is found by multiplying the corresponding columns by the known values of u's or q's. Once the boundary values are obtained, one can compute the interior values using

$$u^i = \int_\Gamma q u^* d\Gamma - \int_\Gamma u q^* d\Gamma, \tag{14}$$

or

$$u^i = \sum_{j=1}^{N} G^{ij} q^j - \sum_{j=1}^{N} \hat{H}^{ij} u^j. \tag{15}$$

In our case we use 7 points on the boundary and 1 interior point. The number of unknowns in this case is 9. One has to solve a system of 9 equations and then evaluate u at one point on each side of each subdomain Ω_i. This step can be done in parallel.

Remarks:

1. If the domain Ω was a rectangle, it is sufficient to take 4 points for the boundary element (one on each side).

2. The boundary element method was applied to inhomogeneous and nonlinear problems (see e.g., [17], [31]).

3. A list of fundamental solutions for various problems is given in Brebbia [8].

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